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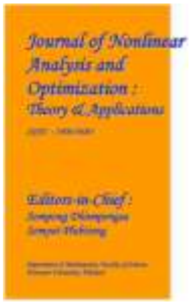


# DYNAMICAL BEHAVIOUR OF A FRACTIONAL-ORDER ECO-EPIDEMIOLOGICAL MODEL WITH HOLLING TYPE II FUNCTIONAL RESPONSE

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**Abstract**
This article examines a model of an order on the fractional of a diseased prey predator. The model has been used in a non-delayed model as a Holling-type II functional response. The eigenvalues of a model are used to test its stability using critical points. Furthermore, the boundedness, uniqueness, existence, and positivity of the solutions have been studied. The locally asymptotically stable has been analyzed using the critical points. The occurrence of Hopf bifurcation has been examined. Finally, the analytical solutions are confirmed through numerical simulations.

**Keywords: Fractional order, Caputo derivatives, Holling type II, Stability, Hopf bifurcation**

**AMS Subject Classification**: 37C75, 92B05.

## 1. Introduction:

Lotka [1] and Volterra's [2] predator-prey models are crucial in current computational ecology, particularly in non-linear differential equation-coupled systems. Functional response is one of the most significant aspects of the prey-predator population, which is why epidemiological models have attracted a lot of interest since Kermack-Mckendrick's pioneering work on SIRS [3]. A generalisation of the classical differentiation and integration of arbitrary orders is fractional calculus. Many researchers are interested in scientific and engineering fields, including biology, fluid dynamics, and medicine [19,22]. The numerous applications of fractional-order calculus have drawn the interest of researchers throughout the last twenty years [5]. Fractional-order biological models have recently attracted the interest of many authors. The main reason lies in the fact that memory-based systems, which exist in a large number of biological systems, are easily relatable to fractional-order models [4]. The fractional-order derivative has the benefit that it allows you to remember the concept of numerical derivative calculation as well as important information about derivative values. Javidi investigated the fractional-order prey-predator model's biological behaviours [11, 23]. This article includes an investigation of the stability of a derivative of a fractional-order model of the mutualistic interaction between two species with infection [21]. Alidousti studied how the capture of predators and scavengers was affected by a prey-predator model with fractional order [10]. Mukherjee et al. investigated the existence, uniqueness, and boundedness of solutions in a restricted region for a predator-prey fractional-order system [14]. The dynamic interaction between prey and predator is one of the most important ecological areas due to its universality and significance.

The dynamical issues involved in the prey-predator mathematical model system can appear easy at first [25, 12]. Mathematical models are essential for understanding, studying, and investigating the expanse and management of infectious diseases [6]. The effects of infectious diseases in a prey-predator system have been extensively studied. In the eco-epidemiological model, the infection of prey is highly significant. Recently, fractional calculations have developed rapidly and displayed a wide range of possible applications [26]. However, due to memory effects, fractional order derivatives in the biological model are more sensible than integer order derivatives. To change ordinary calculus to fractional calculus, it is important to use the Riemann-Lioville and Caputo fractional derivatives. One of the most important processes in any natural ecosystem is the predator-prey model. Caputo introduced the Caputo-type derivative in 1967 [18]. The study investigated a system of fractional order with a Holling type II functional response. The condition for stability of a system in fractional-order, which was developed using Routh-Hurwitz criteria, is that any function that depends on both the current and previous states is a fractional-order derivative [20, 24]. A system with non-linear fractional order stability with the use of the Routh-Hurwitz criterion was investigated by Ahmed et al [13]. Garappa investigated the solution of nonlinear differential equations of fractional order [17]. In a prey-predator model with fractional order, Javidi and Nyamoradi investigated the effects of harvesting. Fractional-order mathematical models are utilised to tackle real-world problems. In dynamical systems, the conversion of the integer-order model into a fractional-order model has shown in popularity. In order to recognize and evaluate the spread and control of infectious diseases, mathematical models are essential.

The novelty of this work is to investigate the prey-predator model's stability analysis through fractional-order derivatives. The analysis demonstrates that fractional calculus is well suited to explain the memory and inherited features of several techniques and materials that are not taken into consideration by classical integer models.

The paper is organized as follows: A mathematical model is developed in Section 2. Section 3 examines the fractional-order dynamical system's preliminary dynamics. Section 4 has evaluated the uniqueness and boundedness of the proposed model. Section 5 examines the stability analysis of the proposed model.The Hopf bifurcation analysis has been studied in Section 6. Numerical simulations of the proposed model are evaluated in Section 7. In Section 8, we conclude the paper and discuss the biological implications of our mathematical results.

## 2. Mathematical Model Formation:

The study presents a three-species predator-prey model, which is further expanded by considering the fractional order derivative. The proposed model becomes:

$$\left.\begin{aligned}
\frac{dU}{dT} &= r_1 U\left(1-\frac{U+V}{K}\right) - \lambda VU - \frac{\alpha_1 UW}{a_1+U},\\
\frac{dV}{dT} &= \lambda VU - D_1 V - \frac{b_1 VW}{a_1+V},\\
\frac{dW}{dT} &= -D_2 W + \frac{Cb_1 VW}{a_1+V} + \frac{C\alpha_1 UW}{a_1+U}\ .
\end{aligned}\right\} \quad (1)$$

subject to positive values $U(0) \geq 0, V(0) \geq 0$ and $W(0) \geq 0$.

**Table 1: Biological representation of system (1) parameters**

| **Parameters** | **Biological Representation** | **Units** |
|---|---|---|
| U | Susceptible prey | Number per unit area (tons) |
| V | Infected prey | Number per unit area (tons) |

| W | Predator | Number per unit area (tons) |
|---|---|---|
| $r_1$ | Intrinsic growth rate of prey | per day $(t^{-1})$ |
| K | Carrying capacity of environment | Number per unit area (tons) |
| $\alpha_1$ | Predation rate of Susceptible prey | per day $(t^{-1})$ |
| $b_1$ | Predation rate of diseased prey | per day $(t^{-1})$ |
| $a_1$ | Half saturation constant | $m$ |
| $C$ | Conversion rate of prey and predator | $0 \leq C \leq 1$ |
| $d_1$ | Ratio of prey death | per day $(t^{-1})$ |
| $d_2$ | Ratio of predator death | per day $(t^{-1})$ |
| $\lambda$ | Rate of infection | per day $(t^{-1})$ |

To reduce system (1) parameters, adjust variables $u = \frac{U}{K}, v = \frac{V}{K}, w = \frac{W}{K}$ and consider dimensionless time $t = \lambda KT$. We now make the following modifications:

$$r = \frac{r_1}{\lambda k}, \alpha = \frac{\alpha_1}{\lambda K}, a = \frac{a_1}{\lambda K}, d_1 = \frac{D_1}{\lambda K}, \theta = \frac{b}{\lambda K}, d_2 = \frac{D_2}{\lambda K}.$$

The transformations listed above can be used to express equation (1) in a non-dimensional form.

$$\left.\begin{aligned} \frac{du}{dt} &= ru(1-u-v) - uv - \frac{\alpha uw}{a+u}, \\ \frac{dv}{dt} &= uv - d_1 v - \frac{\theta vw}{a+v}, \\ \frac{dw}{dt} &= -d_2 w + \frac{c\theta vw}{a+v} + \frac{c\alpha uw}{a+u}. \end{aligned}\right\} \quad (2)$$

subject to positive values $U(0) \geq 0, V(0) \geq 0$ and $W(0) \geq 0$.

After applying the Caputo fractional-order derivative β to system (2), the system is then transformed into the following form:

$$\left.\begin{aligned} \frac{d^\beta u}{dt^\beta} &= ru(1-u-v) - uv - \frac{\alpha uw}{a+u}, \\ \frac{d^\beta v}{dt^\beta} &= uv - d_1 v - \frac{\theta vw}{a+v}, \\ \frac{d^\beta w}{dt^\beta} &= -d_2 w + \frac{c\theta vw}{a+v} + \frac{c\alpha uw}{a+u} \end{aligned}\right\} \quad (3)$$

subject to positive values $U(0) \geq 0, V(0) \geq 0$ and $W(0) \geq 0$.

## 3. Preliminaries:

This section gives basic explanations of fractional differential equations, their significance, and their features. These are essential for proving theorems.

**Definition: 1 [7]**

The Caputo fractional derivative of order $\beta$ is defined as

$$CD_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} f'(s) ds$$

## 4. Existence and Uniqueness of Solutions:

This section examines the existence, uniqueness and boundedness of the solution of the system (3).

**Theorem: 3**

The positive initial conditions of the fractional-order system (3) have a single solution.

**Proof:**

A sufficient requirement for system (3) solutions in the area $\chi \times (0, T]$.

Where,

$$\chi = \{(u, v, w) \in R^3 : \max(|u|, |v|, |w|) \leq \eta\}$$

Now, let us calculate $G(X) = (G_1(X), G_2(X), G_3(X))$

Where,

$$G_1(X) = ru(1-u-v) - uv - \frac{\alpha uw}{a+u},$$

$$\mathrm{G_2(X) = uv - d_1v - \frac{\theta vw}{a+v}},$$
$$\mathrm{G_3(X) = -d_2w + \frac{c\theta vw}{a+v} + \frac{c\alpha uw}{a+u}}.$$
$$\begin{aligned}
\| G(X) - G(\bar{X}) \| &= |G_1(X) - G_1(\bar{X})| + |G_2(X) - G_2(\bar{X})| + |G_3(X) - G_3(\bar{X})| \\
&= \left| \mathrm{ru(1-u-v) - uv - \frac{\alpha uw}{a+u} - r\bar{u}(1-\bar{u}-\bar{v}) + \bar{u}\bar{v} + \frac{\alpha\bar{u}\bar{v}}{a+\bar{u}}} \right| \\
&\quad + \left| \mathrm{uv - \frac{\theta vw}{a+v} - d_1v - \bar{u}\bar{v} + d_1\bar{v} + \frac{\theta\bar{v}\bar{w}}{a+\bar{v}}} \right| \\
&\quad + \left| \mathrm{-d_2w + \frac{c\alpha uw}{a+u} + \frac{c\theta vw}{a+v} + d_2\bar{w} - \frac{c\theta\bar{v}\bar{w}}{a+\bar{v}} + \frac{c\alpha\bar{u}\bar{w}}{a+\bar{u}}} \right| \\
&\le \{r + 2r\eta + \eta + \alpha a(1+c)\eta + \eta(\alpha+1)\}|u-\bar{u}| \\
&\quad + \{r\eta + 2\eta + d_1 + \theta a\eta(1+a) + \theta\eta + \eta(r+2+\theta)\}|v-\bar{v}| \\
&\quad + \{(1+c)a\alpha\eta + (1+c)\theta a\eta + c\eta(\theta+\alpha+d_2)\}|w-\bar{w}| \\
&\le \mathbb{Q}|X-\bar{X}|
\end{aligned}$$

where

$$\mathbb{Q} = \max\{r + 2r\eta + \eta + \alpha a(1+c)\eta + \eta(\alpha+1), r\eta + 2\eta + d_1 + \theta a\eta(1+a) + \theta\eta + \eta(r+2+\theta), (1+c)a\alpha\eta + (1+c)\theta a\eta + c\eta(\theta+\alpha+d_2)\}$$

As a result, the system (3) solution exists and is unique.

**4.1 Boundedness of Solutions:**

**Theorem: 4**

The system's (3) solutions are all positive and uniformly bounded.

**Proof:**

Construct a function

$$V(t) = u(t) + v(t) + \omega w(t)$$

Then for each $\eta > 0$, we obtain

$$\begin{aligned}
{}^{C}D_t^{\beta} + \eta V(t) &= \left(ru - uv - \frac{\alpha uw}{a+u}\right) + \left(uv - d_1v - \frac{\theta vw}{a+v}\right) + \omega\left(-d_2w + \frac{c\theta vw}{a+v} + \frac{c\alpha uv}{a+u}\right) + \eta(u + v + \omega w) \\
&= (r+\eta u) - ru^2 - ruv + c\theta\left(\eta - \frac{1}{c}\right)\frac{vw}{a+v} + \frac{c\alpha wu}{a+u}\left(\eta - \frac{1}{c}\right) + (\eta - d_1)v + \eta(\eta - d_1)w
\end{aligned}$$

By choosing $\eta < \min\{d_1, d_2\}$ and $\eta < \min\left\{\frac{1}{c}\right\}$, we have

$$\begin{aligned}
{}^{C}D^{\beta} + \eta V(t) &\le (r+\eta)u - ru^2 \\
&= (r+\eta)u - ru^2 - \left(\frac{r+\eta}{2}\right)^2 + \left(\frac{r+\eta}{2}\right)^2 \\
&\le \frac{(r+\eta)^2}{2}
\end{aligned}$$

Applying the Lemma 2, we have

$$V(t) \le \left(V(0) - \frac{(r+\eta)^2}{4\eta}\right)E_\beta[\eta t^\beta] + \frac{(r+\eta)^2}{4\eta}$$

Here, we know that $V(t)$ is convergent to $\frac{(r+\eta)^2}{4\eta}$ for $t \to \infty$.

Therefore, the solutions of system (3) with nonnegative initial conditions are all contained within the area $\Omega$.

Where,

$$\Omega = \left\{(u,v,w) \in R_+^3 : V(t) \le \frac{(r+\eta)^2}{4\eta} + \epsilon, \epsilon > 0\right\}.$$

## 5. Critical points and Stability analysis:

The possible critical points of the system (3) are discussed in this section.

(i) $E_0(0,0,0)$ is denoted as a trivial critical point.

(ii) $E_1(1,0,0)$ is denoted as a axial critical point.

(iii) $E_2(\bar{u},0,\bar{w})$ is denoted as a infected free critical point,
Where $\bar{u}=\frac{d_2 a}{d_2-c\alpha}$ and $\bar{w}=\frac{ac((c\alpha-d_2)r-ard_2}{(c\alpha-d_2)^2}$.

(iv) $E_3(\hat{u},\hat{v},0)$ is denoted as a predator free critical point,
Where $\hat{u}=d_1$ and $\hat{v}=\frac{r(1-d_1)}{r+1}$.

(v) $E^*(u^*,v^*,w^*)$ is denoted as a interior critical point,
Where $v^*=\frac{a(ad_2+(d_2-c\alpha)u^*)}{(c\alpha u^*+(c\theta-d_2)(a+u^*))}$ and $w^*=\frac{ac(u^*-d_1)(a+u^*)}{(c\alpha u^*+(c\theta-d_1)(a+u^*))}$,
and $u^*$ is the only positive root of the equation for a quadratic function.

$$Au^2+Bu+C=0,$$

Where $A=r(c\alpha+c\theta-d_2)$,

$$B=(c\theta-d_2)(-r+ar)-r\alpha c+a(d_1+(d_1-c\alpha)r),$$

$$C=-a((r)(c\theta-d_1)+\big(c\alpha(d_1)-ad_2(1+r)\big)).$$

**5.1 Stability Analysis:**

To carry out an investigation of local stability around a number of critical points, we now want to compute the Jacobian matrix.The Jacobian matrix at any $(u,v,w)$ is given by

$$J(u,v,w)=\begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix}$$

Where,

$n_{11}=r(1-2u)-v(r+1)-\frac{\alpha aw}{(a+u)^2}, n_{12}=-u(r+1), n_{13}=-\frac{\alpha u}{a+u}$,

$n_{21}=v, n_{22}=u-d_1-\frac{a\theta w}{(a+v)^2}, n_{23}=\frac{\theta v}{a+v}$,

$n_{31}=\frac{ac\alpha w}{(a+u)^2}, n_{32}=\frac{a\theta wc}{(a+v)^2}, n_{33}=\frac{c\theta v}{(a+v)^2}+\frac{\alpha cu}{a+u}-d_2.$

**Theorem: 5**

$E_0(0,0,0)$ is the trivial critical point, which is saddle.

**Proof:**

The eigenvalues are $\lambda_1=r, \lambda_2=-d_1$ and $\lambda_3=-d_2$.

Thus, $|\arg(\lambda_1)|=0<\frac{\beta\pi}{2}$, $|\arg(\lambda_2)|=\pi>\frac{\beta\pi}{2}$ and $|\arg(\lambda_3)|=\pi>\frac{\beta\pi}{2}$ (see theorem 1 [15]).

Hence, $E_0$ is saddle.

**Theorem : 6**

The axial critical point $E_1(1,0,0)$ , which is unstable.

**Proof:**

The eigenvalues are $\lambda_1=-r, \lambda_2=1-d_1$ and $\lambda_3=-d_2+\frac{c\alpha}{a+u}$.

Thus, $|\arg(\lambda_1)|=0<\frac{\beta\pi}{2}$, $|\arg(\lambda_2)|=\pi>\frac{\beta\pi}{2}$ and $|\arg(\lambda_3)|=\pi>\frac{\beta\pi}{2}$ (see theorem 1 [15]).

Due to numerical simulation table values, $1-d_1$ is positive.

Hence, the equilibrium point $E_1$ is unstable.

**Theorem : 7**

The infected-free critical point $E_2(\bar{u}, 0, \bar{w})$ , which is locally asymptotically stable if $A, C, AB - C$ is positive.

**Proof:**

The Jacobian matrix mentioned above has a characteristic equation that is given by

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0.$$

Where,

$$A = -f_{11} - f_{22},$$
$$B = -f_{31}f_{13} + f_{22}f_{11},$$

$C = f_{13}f_{31}f_{12}$.

Using the Routh Hurwitz criterion (see Proposition 1 [28]),

Hence, $E_2$ is locally asymptotically stable.

**Theorem: 8**

The predator free critical point $E_3(\hat{u}, \hat{v}, 0)$ , which is locally asymptotically stable if $d_2 > c(\alpha + \theta)$.

**Proof:**

The Jacobian matrix mentioned above has a characteristic equation that is given by

$$\lambda^3 + X\lambda^2 + Y\lambda + Z = 0.$$

Where,

$$X = -a_{11} - a_{22},$$
$$Y = -a_{31}a_{13} + a_{22}a_{11},$$

$Z = a_{13}a_{31}a_{12}$.

Using the Routh Hurwitz criterion (see Proposition 1 [28]),

Hence, $E_3$ is locally asymptotically stable.

**Theorem: 9**

The interior critical point $E^*(u^*, v^*, w^*)$, which is locally asymptotically stable.

**Proof:**

The Jacobian matrix mentioned above has a characteristic equation that is given by

$$\lambda^3 + L\lambda^2 + M\lambda + N = 0. \qquad (4)$$

Where,

$$L = -g_{11} - g_{22},$$
$$M = -g_{31}g_{13} + g_{22}g_{11},$$

$N = g_{13}g_{31}g_{12}$.

Using the Routh Hurwitz criterion (see Proposition 1 [28]),

Hence, $E^*$ is locally asymptotically stable.

## 6. Hopf Bifurcation analysis:

In this section, we study the bifurcation of the fractional order model analytically, considering the effect of fractional order $\beta$. The following theorem, which takes the fractional order derivative parameter $\beta$ as a variable, shows the existence of Hopf bifurcation.

**Theorem:11**

When bifurcation parameter β passes through the critical value $\beta^* \in (0,1)$, the fractional-order system (3) undergoes a Hopf bifurcation at the endemic equilibrium point$E^*$, given that the following requirements are met:

(i) the corresponding characteristic equation ($4$) of system ($3$) has a pair of complex conjugates $\lambda_{1,2} = \theta + i\omega$, where $\theta > 0$ and one negative real root $\lambda_3$.

(ii) $m(\beta^*) = \frac{\beta^*\pi}{2} - \min_{1\le i\le 3} |\arg(\lambda_i)| = 0.$

(iii) $\left(\frac{dm(\beta)}{d\beta}\right)_{\beta=\beta^*} \neq 0.$

Here, we provide the conditions in which there would be a Hopf bifurcation at the interior critical point $E^*$ as the derivative's order approaches a critical value.

**Proof:**

Since $\lambda_{1,2} = \theta + i\omega$ and $m(\beta) = \frac{\square\square}{2} - \min_{1 \le i \le 3} |\arg(\lambda_i)|$,
Thus, we get

$$m(\beta^*) = \frac{\beta^* \pi}{2} - \min_{1 \le i \le 3} |\arg(\lambda_i)|$$
$$= \frac{\beta^* \pi}{2} - \arctan\left(\frac{\omega}{\theta}\right)$$
$$= \arctan\left(\frac{\square}{\theta}\right) - \arctan\left(\frac{\omega}{\theta}\right) = 0$$

Furthermore, $\left(\frac{dm(\beta)}{d\beta}\right)_{\beta=\beta^*} = \frac{\pi}{2} \neq 0.$

As a result, when β crosses the crucial value $\beta^*$, the system (3) experiences a Hopf-bifurcation at $E^*$ since all the requirements for its occurrence are met.

## 7. Numerical Simulations:

This section displays the numerical simulation findings for models of fractional-order eco-epidemics with caputo sense. Diethelm et al.'s predictor-corrector approach is used to solve the defined model. [16].

The simulations are executed with the succeeding assumed values:

$$r = 0.7, \alpha = 0.2, a = 0.3, d_1 = 0.4, \theta = 0.4, d_2 = 0.1, c = 0.5$$

(i) When $\beta = 1$, Figures 1 and 2 show the unstable solution for the critical point $E_2(0.7,0.01,0.5)$.

(ii) When $\beta = 0.94$, Figures 3 and 4 show the locally asymptotically stable for the critical point $E_2(0.7,0.01,0.5)$.

Next, the parameter values are chosen as

$$r = 0.5, \alpha = 0.15, a = 0.2, d_1 = 0.1, \theta = 0.4, d_2 = 0.1, c = 0.5$$

(i) Figures 5 and 6 illustrates how the interior critical point $E_4(0.7, 0.04, 0.3)$ becomes unstable when β = 1.

(ii) Figures 7,8 and 9 illustrates the significance of the fractional-order β when $\beta = 0.92, 0.84, 0.76, 0.64$ and the interior critical $E_4(0.7, 0.04, 0.3)$ becomes locally asymptotically stable.

Here, we observe that the infected prey species and the susceptible prey-predator species begin to oscillate in density. However, this high-amplitude species oscillation causes an exceptionally low population density, which could cause the multispecies community to become unstable and raise the probability that some species will go extinct. From Figure 7, 8, 9, we can conclude that the fractional-order derivative values decreased from 1 to 0.92, 0.84, 0.76, and 0.64 the equilibrium point is transformed from unstable state to stable state. As a result, we can say that the integer-order model is less stable than the fractional-order model. Hopf-bifurcation is used to shift the dynamics from an unstable to a stable steady state when the fractional order derivative parameter β exceeds the threshold value of $\beta^* = 0.41$. Hopf bifurcation occurs at fractional order derivative $\beta = 0.41$ is shown in Figures (10, 11, 12). According to the numerical simulation results, when the value of β (fractional derivative order) increases from 0 to 1, the proposed system stabilises the system, turning it from an unstable to a stable state.

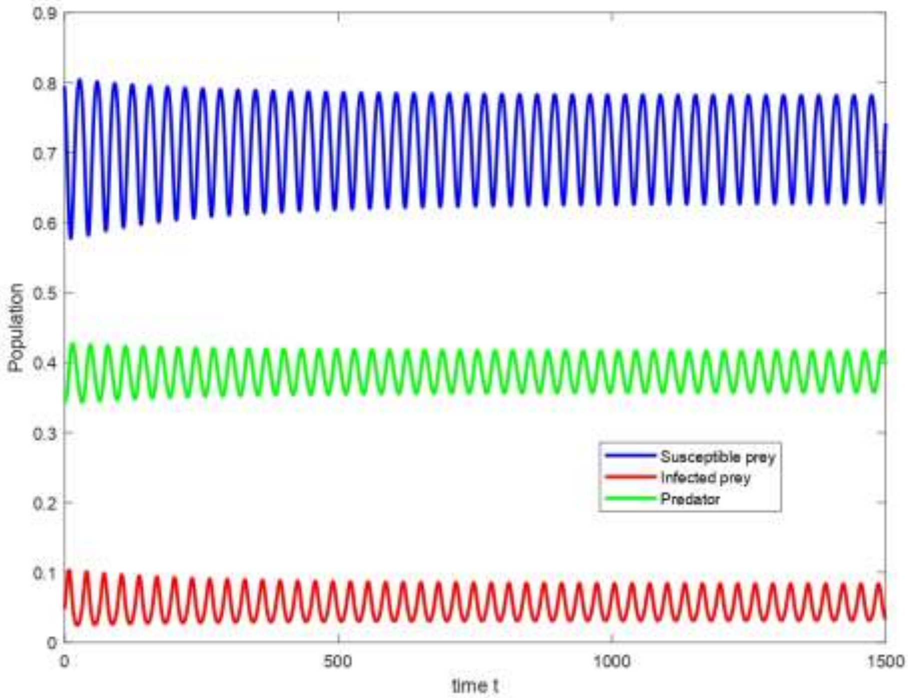


**Figure 1:** Time series solution of system (3) of the critical point $E_2$ when $\beta = 1$.

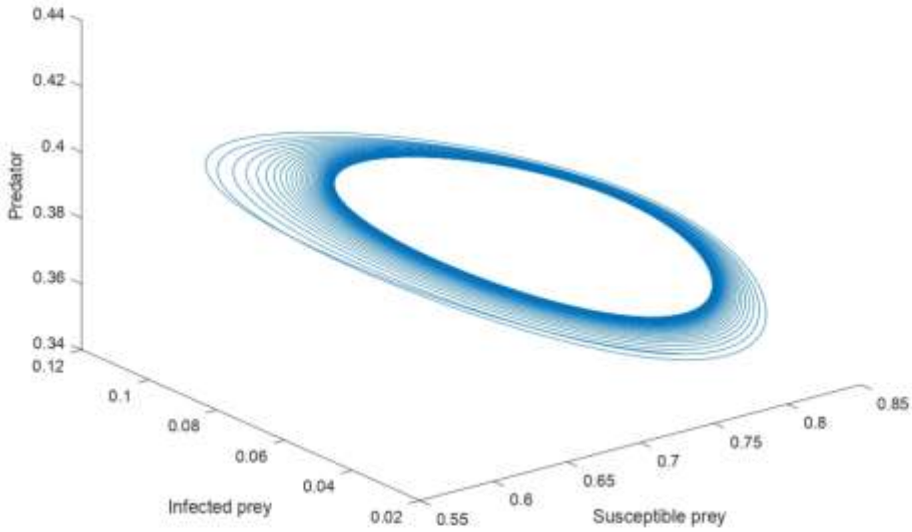


**Figure 2:** At the critical point $E_2$, the phase portrait solution of system (3) when $\beta = 1$.

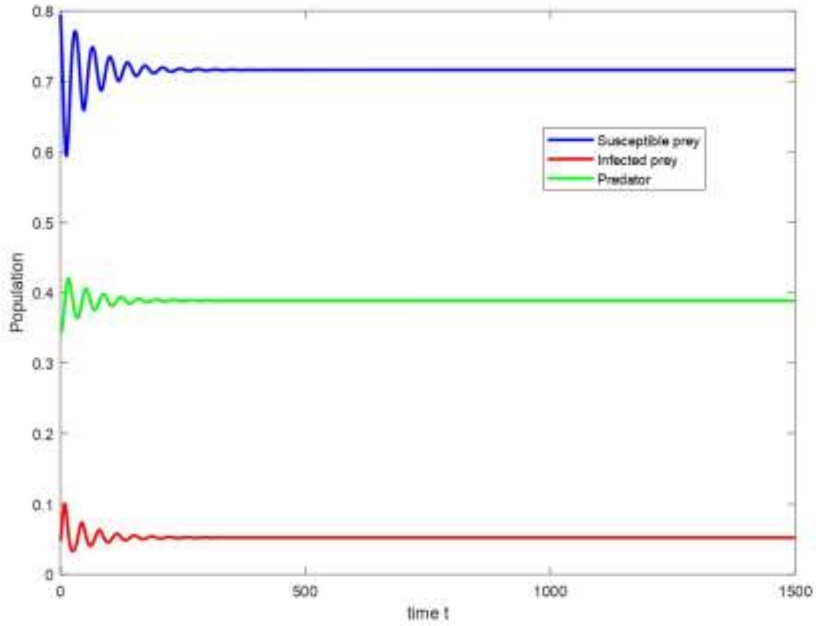


**Figure 3:** Time series solution of system (3) of the critical point $E_2$ when $\beta = 0.94$.

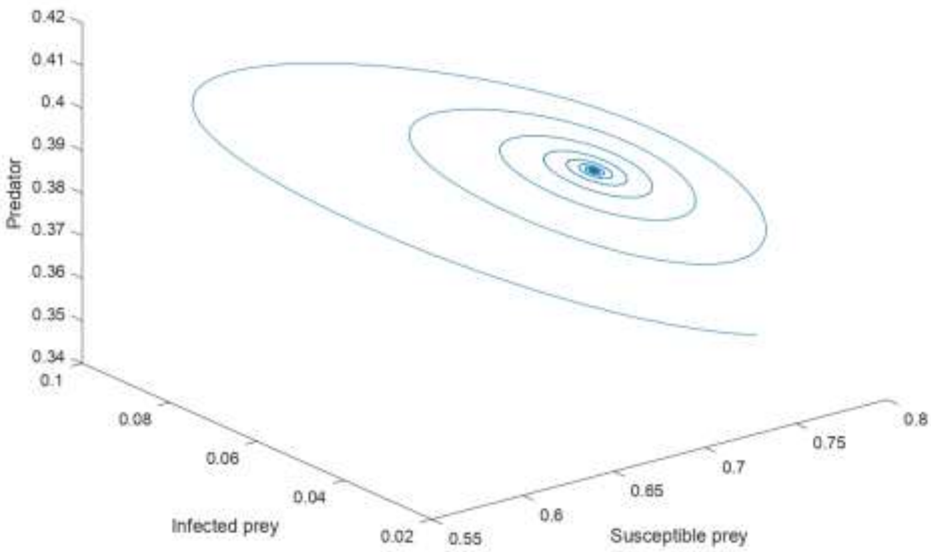


**Figure 4:** At the critical point $E_2$, the phase portrait solution of system (3) when $\beta = 0.94$

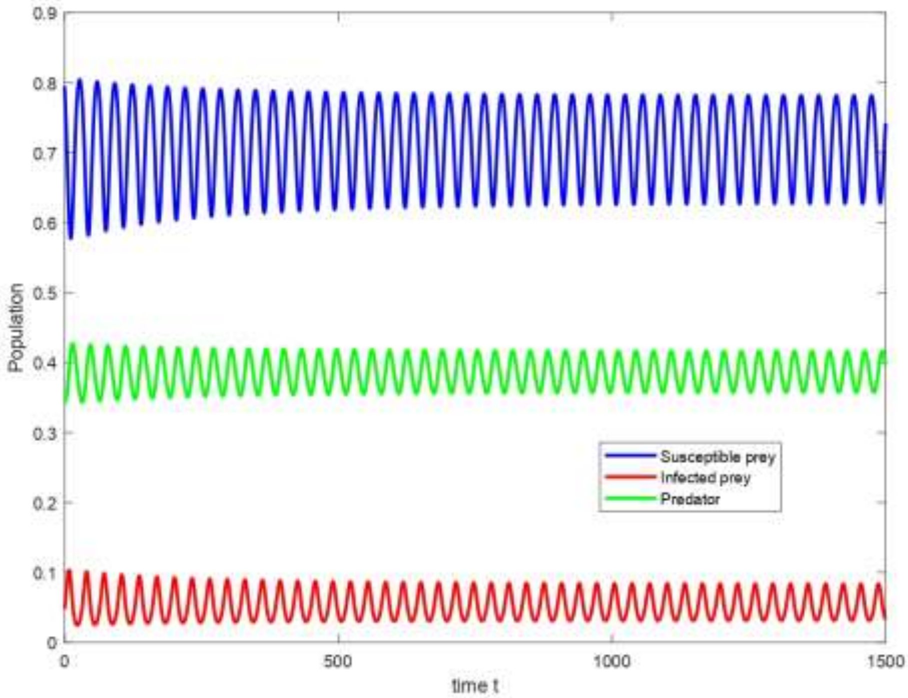


**Figure 5:** Time series solution of system (3) of the critical point $E_4$ when $\beta = 1$.

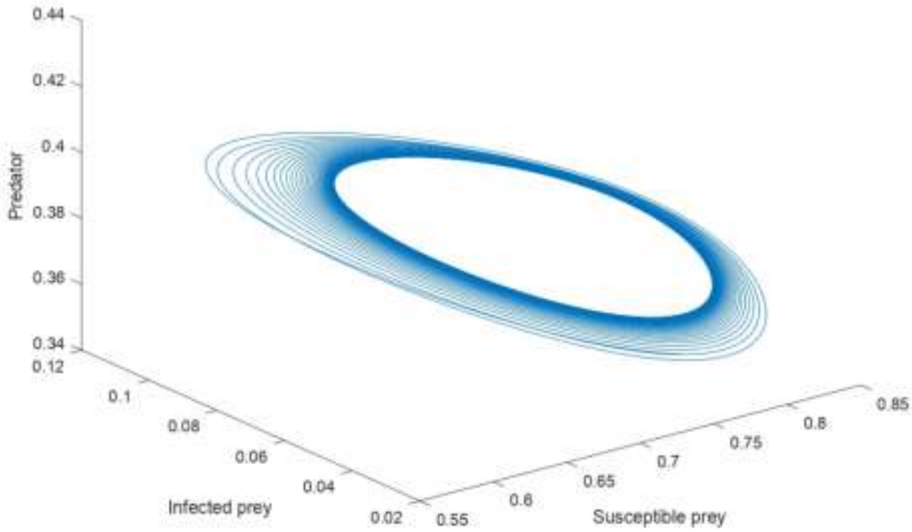


**Figure 6:** At the critical point $E_4$, the phase portrait solution of system (3) when $\beta = 1$.

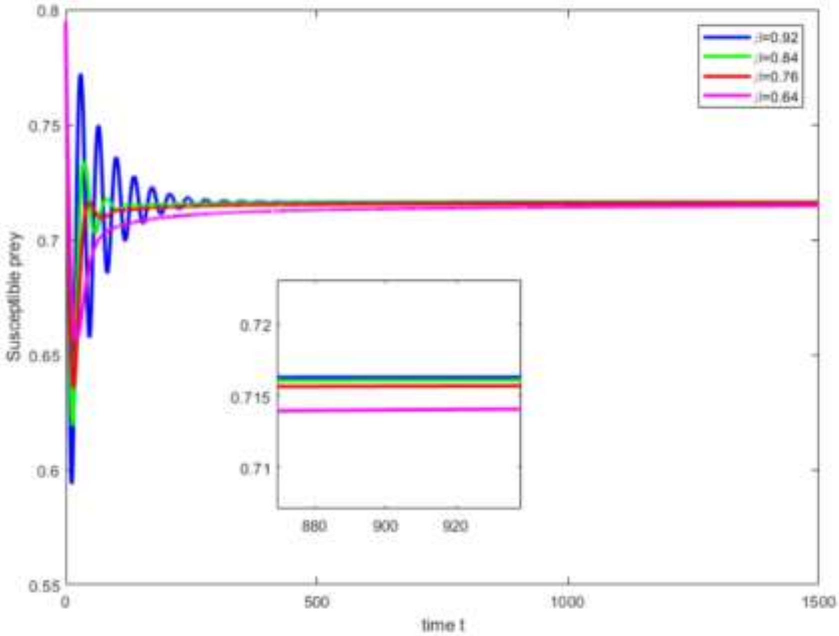


**Figure 7:** Susceptible prey population response to the fractional-order β at system (3) critical point $E_4$.

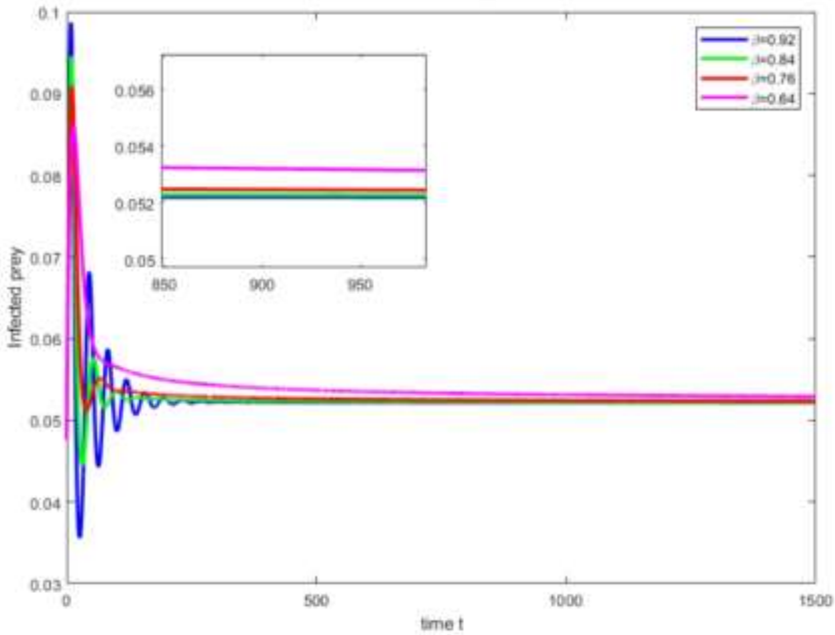


**Figure 8:** Infected prey population response to the fractional-order β at system (3) critical point $E_4$.

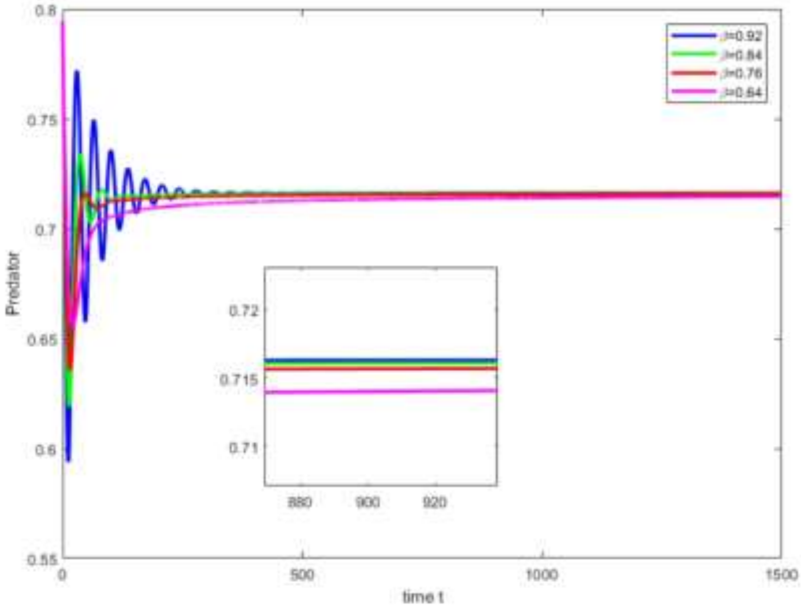


**Figure 9:** Predator population response to the fractional-order β at system (3) critical point $E_4$.

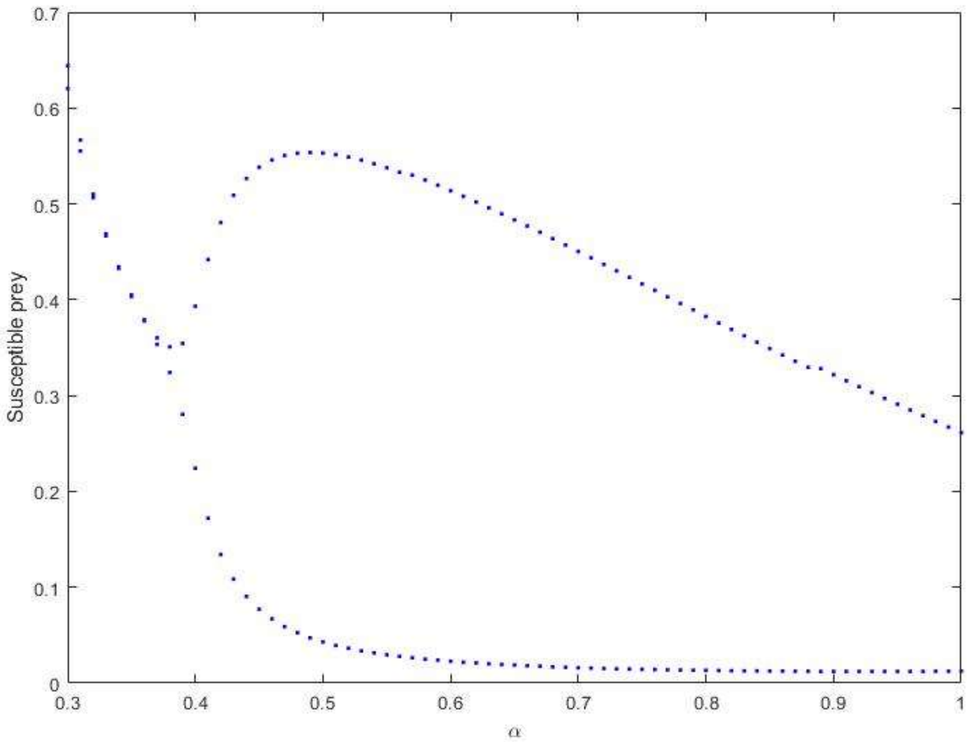


**Figure 10:** Bifurcation diagram for the fractional-order β on the susceptible prey population at system (3) critical point$E_4$.

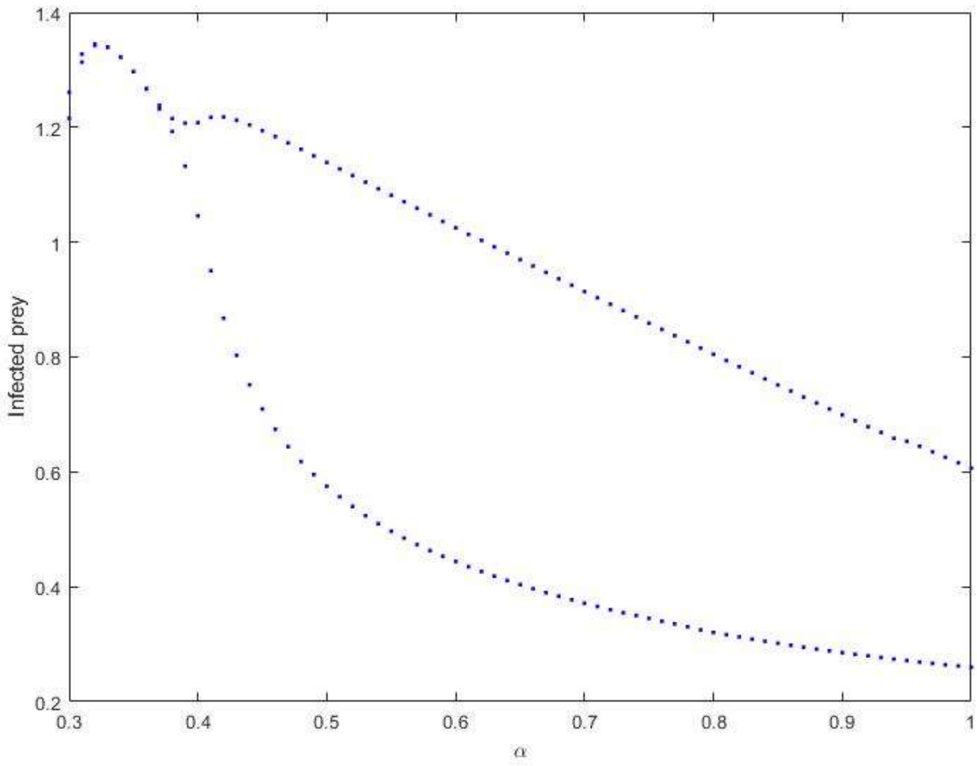


**Figure 11:** Bifurcation diagram for the fractional-order β on the infected prey population at system (3) critical point$E_4$.

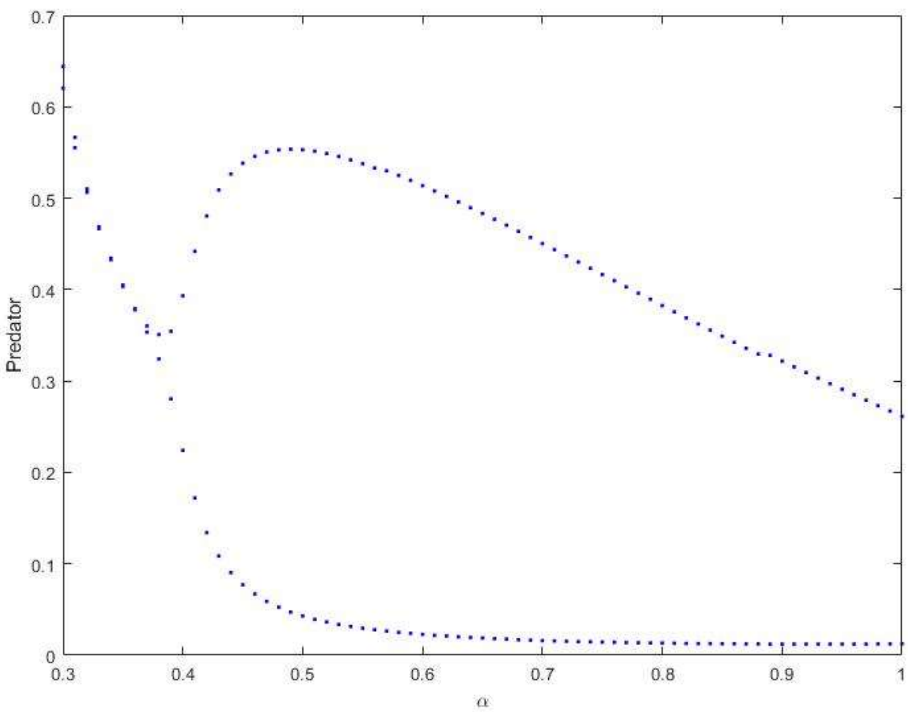


**Figure 12:** Bifurcation diagram for the fractional-order β on the predator population at system (3) critical point$E_4$.

**8. Conclusion:**

This study analysed a three-species food web model using fractional-order derivatives. We have also looked into the local stability of each critical point in our proposed fractional-order system. A number of biological systems that are highly dependent on historical events have been described using fractional-order mathematical models. These findings suggest that the mathematical model of fractional order can be useful in explaining system dynamics with useful memory. For a fractional-order system, the unstable system with integer-order $\beta = 1$ becomes stable at various values of $\beta$ in the range $0 < \beta < 1$. Hence, the fractional-order derivative $\beta$ provides detailed information about the proposed model's dynamical behaviour.

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